On the Construction of the Implicit Finite Difference Schemes for Solving a Time-Space Fractional Convection-Diffusion Equation

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Abstract:

A new approach of the finite difference method is implemented for solving time-space fractional convection-diffusion equations. This technique converts the issue to the implicit scheme by using the Laplace algorithm type-1 and Grünwald-Letnikov formula in handling the time derivative and space derivative respectively. The novel impact of this paper is to extend the finite difference method involving fractional time derivatives. The proposed implicit scheme is $(2-\gamma)$ order accuracy in time and second-order accuracy in space. Moreover, stability and convergence analyses are proved. Our numerical examples show the behavior of the solution for varying values of fractional derivatives.

Key words: Finite difference method; Caputo fractional derivative; Riemann-Liouville fractional integral; Grünwald-Letnikov formula.

1. Introduction

Many phenomena such as earthquakes and non-Markov Processes can be represented by diffusion and convection processes. Through these processes, the physical quantities are transferred, and can be modeled this phenomenon by convection-diffusion equations (CDE). Recently, fractional calculus has proven helpful in the development of different models for fractional diffusion by linking fractional constitutive laws, fractional Brownian motions, and random walk, for more details, see (Henry, Langlands and Straka, 2010), (Metzler and Klafter, 2000) and (Mandelbrot and Van Ness, 1968).

In the last decades, various methods have been introduced for finding a reliable approximate technique for solving a class of diffusion equations. Sayevand et al (Sayevand and Arjang, 2016) used a finite volume element method for finding a numerical scheme that approximates the sub-diffusion equation. Jia and Wang (Jia and Wang, 2016) applied a volume-penalization to approximate the fractional diffusion equations. While in (Zhao *et al.*, 2016) Zhao et al used spatial quasi-Wilson nonconforming to convert a time-fractional diffusion equation to an implicit scheme. Izadkhah et al used the Lagrange polynomials and Gegenbauer polynomials for handling the derivatives that appear in CDE (Izadkhah and Saberi-Nadjafi, 2015). While Saadatmandi et al (Saadatmandi, Dehghan and Azizi, 2012) solved fractional CDEs by improving sinc–legendre collocation method.

In 2021 Sene discussed the stability and convergence analysis of the numerical solution for the fractional differential diffusion equations with reaction terms (Sene, 2021a). Also, (Sene, 2021b) Sene introduced some applications for a fractional-order system described by Caputo fractional derivatives. In (Qu, She and Liu, 2021) Qu et al studied the fractional diffusion equations with integral fractional Laplacian by recasting the equation into an equivalent Ritz formulation. Shen et al (Shen, Li and Shao, 2020) proposed the implicit finite difference scheme and generated a linear system with a real Toeplitz structure. Mishra (Mishra and Dubey, 2020) construct a series solution for the space fractional diffusion equations with conformable derivatives.

In (Edwan *et al.*, 2021) Edwan et al introduced a finite volume method and finite difference method with supporting analysis for solving a space-fractional convection-diffusion equation, the extension of this work is applied for solving a time-space fractional convection-diffusion equation by using the finite difference method.

This paper aims to improve a finite difference method (FDM) in finding approximate solutions for a time-space fractional convection-diffusion equation (TSFCDE) in the following form:

$$\frac{\partial^{\gamma}\phi(x,t)}{\partial t^{\gamma}} + \epsilon \frac{\partial}{\partial x} J_a^{\alpha}\phi(x,t) - v\frac{\partial^2\phi(x,t)}{\partial x^2} = g(x,t),\ t \geq 0,\ 0 < \gamma \leq 1,\ ,0 \leq \alpha < 1, \quad (1)$$

subject to the initial condition

$$\phi(x,0) = f(x) \qquad a \leq x \leq b \quad (2)$$

ϵ and v are positive parameters, γ is the order of the time-fractional, α is the order of space fractional integral, $g(x,t), f(x)$ are a smooth function, and $\phi(x,t)$ is concentration (Edwan *et al.*, 2021),(Izadkhah and Saberi-Nadjafi, 2015).

The organization of this paper is as follows: Some preliminary about fractional derivatives and integrals are proposed in section 2. Section 3 contains a new formulation of FDM for solving the initial value problem IVP given in Eq. (1) and Eq. (2), supported with theoretical analysis. Numerical examples are given in Section 4 to show the validity of the method. Finally, in Section 5 some conclusions are given.

2. Basic Definitions

In this section, some preliminary definitions and theorems related to fractional calculus are introduced briefly, such as the Grünwald-Letnikov formula, the Riemann-Liouville fractional integral, and the Caputo fractional derivative.

Definition I. (see (Podlubny, 1998)) The fractional integral of order $\alpha > 0$ in Riemann-Liouville sense $J_a^{\alpha}\phi(x)$ is defined as:

$$J_a^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\phi(t)dt \text{ , provided that } \phi \in L_1[a,b], \quad (3)$$

where $\phi \in L_1[a,b], \phi\colon [a,b] \to \mathbb{C}$, ϕ is measurable on [a, b] and $\int\ |\phi(x)|\,dx\ <\ \infty$.

Definition II. (see (Podlubny, 1998)) Let $\phi \in C^{\lceil\alpha\rceil}[a,b], \alpha > 0$. Then the fractional derivative of order α in Grünwald-Letnikov sense $\widetilde{\mathcal{D}}_a^{\alpha}\phi(x)$ is defined as:

$$\widetilde{\mathcal{D}}_a^{\alpha}\phi(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\left[\frac{x-a}{h}\right]}(-1)^k\binom{\alpha}{k}\phi(x-kh),\ a < x \leq b,\ h = \frac{x-a}{N}, \quad (4)$$

where $C^{\lceil\alpha\rceil}[a,b]$ the set of functions with continuous $\lceil\alpha\rceil^{th}$derivative on $[a,b]$.

Theorem I. Let $\alpha > 0$, and $\phi \in C[a,b]$, then

$$J_a^{\alpha}\phi(x) = \lim_{h\to 0} h^{\alpha}\sum_{k=0}^{\left[\frac{x-a}{h}\right]} w_k^{\alpha}\phi(x-kh),\ h = \frac{x-a}{N}\ , a < x \leq b, \quad (5)$$

where $w_0^{\alpha} = 1,\ w_1^{\alpha} = \alpha$ and $w_k^{\alpha} = \left(1 - \frac{(1-\alpha)}{k}\right)w_{k-1}^{\alpha}, k = 2,3,\ldots,$

where $C[a,b]$ *the set of functions on* $[a,b]$.

Proof of theorem I. see (Arqub *et al.*, 2020).

Definition III. (see (Podlubny, 1998)) Let $\phi\ \in L_1[a,b], \gamma > 0$ then the fractional derivative $\mathcal{D}_*^{\gamma}\phi(x)$ of order γ in Caputo sense defined by

$$\mathcal{D}_*^{\gamma}\phi(x) = \begin{cases} \frac{1}{\Gamma(n-\gamma)}\int_0^x \frac{\phi^{(n)}(s)}{(x-s)^{\gamma+1-n}}ds & ,n-1<\gamma<n \\ \frac{d^n\phi(x)}{dx^n}, & \gamma = n. \end{cases}, \text{ where } n = \lceil \alpha \rceil, \tag{6}$$

where $\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1}e^{-s}ds,\ \alpha > 0, \Gamma(\alpha+1) = \alpha\Gamma(\alpha), \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{sin\pi\alpha}$.

3. Finite difference method

A new formulation of FDM is introduced for solving a TSFCDE in the following form:

$$\mathcal{D}_*^{\gamma}\phi(x,t) + \epsilon\frac{\partial}{\partial x}J_a^{\alpha}\phi(x,t) - v\frac{\partial^2\phi(x,t)}{\partial x^2} = g(x,t). \quad t>0\ ,0\le\alpha<1,\ 0<\gamma\le 1, \tag{7}$$

subject to the initial condition

$$\phi(x,0) = f(x) \qquad a \le x \le b, \tag{8}$$

where $\mathcal{D}_*^{\gamma}\phi(x,t)$ is the Caputo fractional derivative with respect to t, $J_a^{\alpha}\phi(x,t)$ is fractional integral with respect to x in Riemann-Liouville sense, and $g(x,t), f(x)$ are smooth functions.

First, discretize a finite domain $\mathrm{I} = [a,b]$ and that yields nodes $x_i = a + ih, h = \frac{b-a}{N}, i = 0,1,\dots,N$, and define a temporal partition $t_n = n\tau$ where τ is the time step $,n = 0,1,\dots$. Use the difference formulas given in Eq. (9) and Eq. (10) for evaluating the first and second derivative in Eq. (7) at $x = x_i$, and the Grünwald-Letnikov formula given in Eq. (11) for approximate the Riemann-Liouville integral $J_a^{\alpha}\phi(x,t)$ at $x = x_{i+1}$ and $x = x_{i-1}$.

$$\left.\frac{\partial\phi(x,t)}{\partial x}\right|_{x=x_i} = \frac{\phi(x_{i+1},t)-\phi(x_{i-1},t)}{2h} + \mathcal{O}(h^2), \tag{9}$$

$$\left.\frac{\partial^2\phi(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{\phi(x_{i-1},t)-2\phi(x_i,t)+\phi(x_{i+1},t)}{h^2} + \mathcal{O}(h^2), \tag{10}$$

$$J_a^{\alpha}\phi(x,t) = h^{\alpha}\sum_{j=0}^{N} w_j^{\alpha}\phi(x-jh,t) + o(1), \tag{11}$$

then Eq. (7) at (x_i, t_{n+1}) become

$$\begin{aligned} \mathcal{D}_*^{\gamma}\phi(x_i,t)\big|_{t=t_{n+1}} = \\ &-\frac{\epsilon}{2h}\left[h^{\alpha}\sum_{j=0}^{i+1} w_j^{\alpha}\phi(x_{i-j+1},t_{n+1}) + o(1) - h^{\alpha}\sum_{j=0}^{i-1} w_j^{\alpha}\phi(x_{i-j-1},t_{n+1}) + o(1)\right] \\ &+v\left[\frac{\phi(x_{i-1},t_{n+1})-2\phi(x_i,t_{n+1})+\phi(x_{i+1},t_{n+1})}{h^2} + \mathcal{O}(h^2)\right] + g(x_i,t_{n+1}). \end{aligned} \tag{12}$$

Now, use the Laplace algorithm type-1 (L1 –algorithm) for discretize the time derivative in Caputo sense $0<\gamma<1$, see (Oldham and Spanier, 1974).

$$\mathcal{D}_*^{\gamma}\phi(x_i,t_{n+1}) = \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\sum_{r=0}^{n} b_r^{\gamma}[\phi(x_i,t_{n+1-r}) - \phi(x_i,t_{n-r})] + \mathcal{O}(\tau^{2-\gamma}), \tag{13}$$

where $b_r^{\gamma} = (r+1)^{1-\gamma} - r^{1-\gamma}, r = 0,1,\dots,n$.

Letting $\phi_i^n \approx \phi(x_i,t_n)$ denote the numerical solution, we have

$$\begin{aligned} \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\sum_{r=0}^{n} b_r^{\gamma}\left[\phi_i^{n+1-r} - \phi_i^{n-r}\right] = -\frac{\epsilon}{2h}\left[h^{\alpha}\sum_{j=0}^{i+1} w_j^{\alpha}\phi_{i-j+1}^{n+1} - h^{\alpha}\sum_{j=0}^{i-1} w_j^{\alpha}\phi_{i-j-1}^{n+1}\right] \\ +v\left[\frac{\phi_{i-1}^{n+1}-2\phi_i^{n+1}+\phi_{i+1}^{n+1}}{h^2}\right] + g_i^{n+1}. \end{aligned} \tag{14}$$

Rewrite Eq. (14) as:

$$\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\sum_{r=0}^{n} b_r^{\gamma}\left[\phi_i^{n+1-r}-\phi_i^{n-r}\right] = -\sum_{j=0}^{N} a_{ij}\phi_j^{n+1}+g_i^{n+1}, \tag{15}$$

where
$$a_{ij}=\begin{cases}\frac{\epsilon h^{\alpha-1}[w_{i-j+1}^{\alpha}-w_{i-j-1}^{\alpha}]}{2} & ,j<i-1\\ \frac{\epsilon h^{\alpha-1}[w_{2}^{\alpha}-w_{0}^{\alpha}]}{2}-\frac{v}{h^2} & ,j=i-1\\ \frac{\epsilon h^{\alpha-1}w_{1}^{\alpha}}{2}+\frac{2v}{h^2} & ,j=i\\ \frac{\epsilon h^{\alpha-1}w_{0}^{\alpha}}{2}-\frac{v}{h^2} & ,j=i+1\\ 0 & ,j>i+1\end{cases} \tag{16}$$

By denoting the solution vector $\phi^n=[\phi_0^n,\phi_1^n,\dots,\phi_N^n]$ and source vector $g^{n+1}=[g(x_0,t_{n+1}),g(x_1,t_{n+1}),\dots,g(x_N,t_{n+1})]$, then the vector equation given by

$$(I+\Gamma(2-\gamma)\tau^{\gamma}S)\phi^{n+1}=b_n^{\gamma}\phi^0+\sum_{r=0}^{n-1}(b_r^{\gamma}-b_{r+1}^{\gamma})\phi^{n-r}+\tau^{\gamma}\Gamma(2-\gamma)g^{n+1}, \tag{17}$$

where the matrix S has elements $s_{ij}=a_{ij}$.

For $0\le\alpha<1$ if $v>\frac{\epsilon h^{\alpha+1}}{2}$ then the diagonal elements in $C=I+\Gamma(2-\gamma)\tau^{\gamma}S$ are positive and C is strictly diagonally dominant. Moreover, the matrix $C^{-1}=(I+\Gamma(2-\gamma)\tau^{\gamma}S)^{-1}$ exists and the spectral radius $\rho(C)<1$, hence the implicit scheme (17) is conditionally stable. For more details see (Edwan *et al.*, 2021). L1 –algorithm and Eq. (12) guarantee the consistency of the implicit scheme (17) with $(2-\gamma)$ order accuracy in time and second-order accuracy in space.

4. Numerical Experiments

Several examples are introduced in this section to evince the accuracy of the proposed method, For $\gamma=1$ the given IVP (1) and (2) convert to SFCDE, this issue has been discussed and many examples have been introduced in (Edwan *et al.*, 2021). The FDM works well for solving this problem, in this approach, we can select randomly the space-fractional derivative, time-fractional derivative, and nonlinear initial condition. The computations are performed by Wolfram Mathematica software 11.

Example 4.1: Consider the following TSFCDE

$$\mathcal{D}_*^{\gamma}\phi(x,t)+\epsilon\frac{\partial}{\partial x}J_a^{\alpha}\phi(x,t)-v\frac{\partial^2\phi(x,t)}{\partial x^2}=g(x,t).\quad t>0\ ,0\le\alpha<1,\ 0<\gamma\le1, \tag{18}$$

$x\in[0,1.2],\alpha=0.15,\epsilon=1,v=2,\ g(x,t)=0$ subject to the initial condition $\phi(x,0)=Exp(mx), m=1.17712434446770$. The numerical results of FDM at $\alpha=0.15$ with varying γ such that $\gamma\in\{0.75,0.85,0.95,1\}$ are given in Table1, at the time $t=0.5$ and $x\in[0,1]$ using $\tau=0.025$ and $h=0.0625$. In figure 1 the solutions behavior of $\phi(x,t)$ are given at $\alpha=0.15, x\in[0,1.2]$. varying γ such that $\gamma\in\{0.75,0.85,0.95,1\}$.

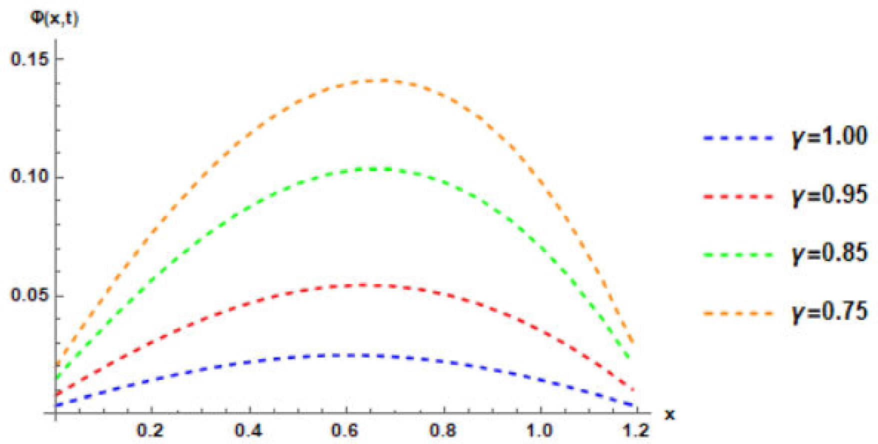

Figure 1. The solutions behavior of $\phi(x,t)$ for Example 4.1 at $\alpha = 0.15, t = 0.5$ with varying γ.

Table 1. The numerical results of $\phi(x,t)$ for Example 4.1 at $\alpha = 0.15, t = 0.5$ with varying γ

x	$\gamma = 0.75$	$\gamma = 0.85$	$\gamma = 0.95$	$\gamma = 1.00$
0.0	0.0210271	0.0157564	0.00880711	0.00457635
0.0625	0.0410225	0.0308125	0.0173037	0.00905871
0.125	0.0598439	0.0450291	0.0253746	0.0133547
0.1875	0.0773284	0.0582548	0.0329	0.0173714
0.25	0.0933044	0.0703346	0.0397618	0.0210204
0.3125	0.107596	0.081114	0.0458471	0.0242196
0.375	0.120025	0.0904414	0.0510501	0.0268961
0.4375	0.130411	0.0981696	0.0552741	0.0289877
0.5	0.138575	0.104157	0.0584332	0.0304446
0.5625	0.144338	0.108269	0.0604531	0.0312305
0.625	0.147523	0.110379	0.0612727	0.0313239
0.6875	0.147953	0.110367	0.0608444	0.0307182
0.75	0.145454	0.108125	0.0591349	0.0294223
0.8125	0.139855	0.103553	0.0561256	0.0274605
0.875	0.130986	0.09656	0.051812	0.024872
0.9375	0.118682	0.0870671	0.0462041	0.0217101
1.0	0.102779	0.0750042	0.0393255	0.0180413

Example 4.2: Consider the following TSFCDE

$$\mathcal{D}_*^{\gamma}\phi(x,t) + \epsilon\frac{\partial}{\partial x}J_a^{\alpha}\phi(x,t) - v\frac{\partial^2\phi(x,t)}{\partial x^2} = g(x,t). \quad t > 0\ ,0 \leq \alpha < 1,\ 0 < \gamma \leq 1, \tag{19}$$

$x \in [0,2.25], \alpha = 0.15, \epsilon = 1, v = 2,\ g(x,t) = 0$ subject to the initial condition $\phi(x,0) = -sin\ (\pi x)$. The numerical results of FDM at $\alpha = 0.15$ with varying γ such that $\gamma \in \{0.75,0.85,0.95,1\}$ are given in Table 2, at the time $t = 0.5$ and $x \in [0,2]$ using $\tau = 0.025$ and $h = 0.125$. In figure 2 the solutions behavior of $\phi(x,t)$ are given at $\alpha = 0.15, x \in [0,2.25]$. varying γ such that $\gamma \in \{0.75,0.85,0.95,1\}$.

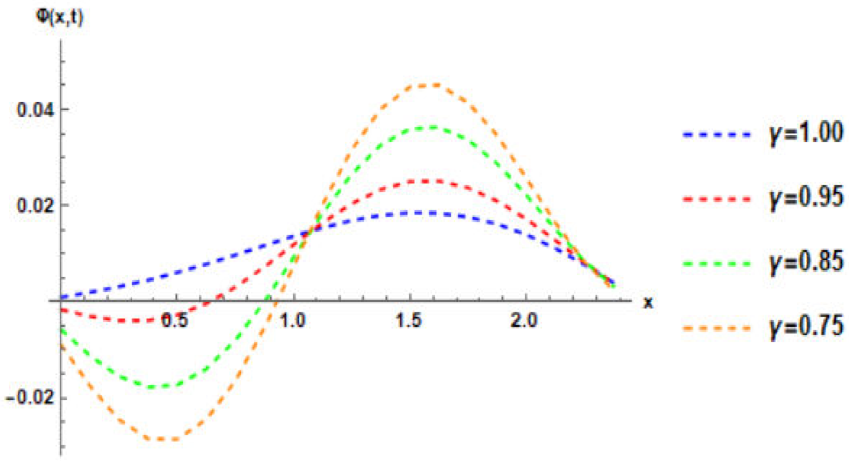

Figure 2. The solutions behavior of $\phi(x,t)$of Example 4.2 at $\alpha = 0.15, t = 0.5$ with varying γ.

Table 2. The numerical results of $\phi(x,t)$ for Example 4.2 at $\alpha = 0.15, t = 0.5$ with varying γ

x	$\gamma = 0.75$	$\gamma = 0.85$	$\gamma = 0.95$	$\gamma = 1.00$
0.0	-0.00923358	-0.00606334	-0.00185397	0.00073433
0.125	-0.0183269	-0.0119511	-0.00355519	0.00157357
0.25	-0.025664	-0.016563	-0.00464082	0.00260969
0.375	-0.0298581	-0.0189596	-0.00472374	0.00390936
0.5	-0.0299729	-0.0185123	-0.00355345	0.00550534
0.625	-0.0256698	-0.0150033	-0.00105676	0.00739041
0.75	-0.0172642	-0.0086638	0.00264675	0.00951492
0.875	-0.00568183	-0.000144151	0.00725695	0.0117885
1.0	0.00767731	0.00957843	0.0123313	0.014086
1.125	0.0211469	0.0193416	0.017343	0.0162565
1.25	0.0330422	0.0279669	0.0217516	0.0181354

1.375	0.0419107	0.034431	0.0250744	0.0195587
1.5	0.0467472	0.0380128	0.0269499	0.0203762
1.625	0.0471396	0.0383944	0.0271833	0.0204658
1.75	0.0433257	0.035702	0.0257674	0.0197442
1.875	0.0361498	0.0304781	0.0228779	0.0181759
2.0	0.0269279	0.0235928	0.0188424	0.0157779

Example 4.3. Consider the following TSFCDE

$$\mathcal{D}_*^{\gamma}\phi(x,t) + \epsilon\frac{\partial}{\partial x}J_a^{\alpha}\phi(x,t) - v\frac{\partial^2\phi(x,t)}{\partial x^2} = g(x,t). \quad t > 0\ ,0 \le \alpha < 1,\ 0 < \gamma \le 1, \tag{20}$$

$x \in [0,1.2], \alpha = 0.15, \epsilon = 0.1, v = 0.02,\ g(x,t) = x^2$ subject to the initial condition $\phi(x,0) = Exp(mx), m = 1.17712434446770$. The numerical results of FDM at $\alpha = 0.15$ with varying γ such that $\gamma \in \{0.75,0.85,0.95,1\}$ are given in Table 3, at the time $t = 0.5$ and $x \in [0,1]$ using $\tau = 0.025$ and $h = 0.0625$. In figure 3 the solutions behavior of $\phi(x,t)$ are given at $\alpha = 0.15, x \in [0,1.2]$. varying γ such that $\gamma \in \{0.75,0.85,0.95,1\}$.

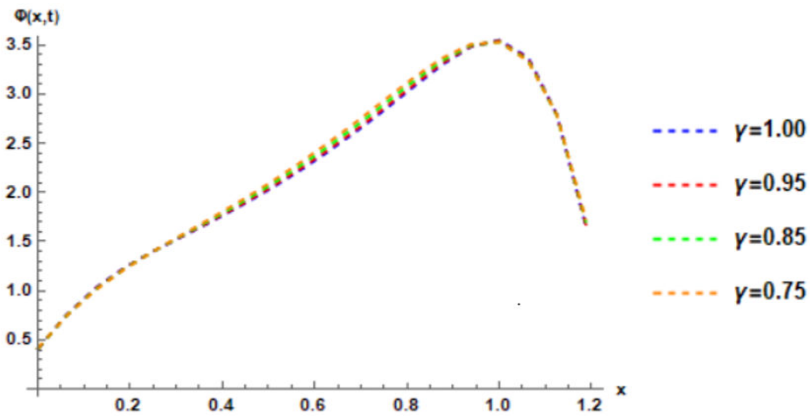

Figure 3. The solutions behavior of $\phi(x,t)$of Example 4.3 at $\alpha = 0.15, t = 0.5$ with varying γ.

Table 3. The numerical results of $\phi(x,t)$ for Example 4.3 at $\alpha = 0.15, t = 0.5$ with varying γ.

x	$\gamma = 0.75$	$\gamma = 0.85$	$\gamma = 0.95$	$\gamma = 1.00$
0.0	0.422426	0.419693	0.416903	0.415494
0.0625	0.753864	0.75772	0.763718	0.767613
0.125	1.01279	1.02011	1.0303	1.03642
0.1875	1.22151	1.2267	1.23386	1.23794
0.25	1.40074	1.40017	1.40043	1.40063
0.3125	1.56672	1.5591	1.55182	1.5482
0.375	1.73074	1.71615	1.7021	1.69532
0.4375	1.90009	1.87907	1.8591	1.84966
0.5	2.07919	2.05225	2.0269	2.01505
0.5625	2.27051	2.23816	2.20768	2.19346
0.625	2.47497	2.43794	2.40263	2.38605
0.6875	2.69175	2.65143	2.61211	2.59336
0.75	2.91707	2.87612	2.83487	2.81469
0.8125	3.14175	3.1046	3.06568	3.04593
0.875	3.34673	3.31936	3.28975	3.27401
0.9375	3.49603	3.48415	3.47208	3.4654
1.0	3.52745	3.53212	3.54153	3.54746

5. Conclusions

The FDM has been developed for solving the TSFCDE, the numerical solution has been represented in an implicit scheme, in this method L1 –algorithm has been used for approximating the time-Caputo fractional derivative, and the fractional Grünwald-Letnikov formula for approximate the Riemann-Liouville integral $J_a^\alpha \phi(x,t)$ and yield an implicit scheme that is with $(2-\gamma)$ order accuracy in time and second-order accuracy in space. Several examples have been introduced and the numerical results have shown this method can solve the problem effectively. The calculations have been performed by using the Wolfram Mathematica 11. In the future, we can extend this method to solve TSFCDE with variable coefficients.

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